An alternative classical approach to the quantum-mechanical definition of the scattering cross section

J. Fiol, R. G. Pregliasco,[a] I. Samengo,[a] and R. O. Barrachina[a]
Centro Atómico Bariloche and Instituto Balseiro,[b] 8400 S. C. de Bariloche, Río Negro, Argentina

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We discuss an alternative classical description of the nonrelativistic scattering process that is completely analogous to the time-independent formalism employed in quantum-mechanics textbooks. This approach provides an elementary introduction to the standard stationary scattering theory, embodying many of its essential concepts. © *1997 American Association of Physics Teachers.*

I. INTRODUCTION

Quantum mechanics has always represented a tough challenge to the student who meets those new concepts for the first time. In order to help in their visualization and understanding, it is a usual practice to resort to classical images of quantum-mechanical effects. In this sense, the scattering process represents a specially conflicting subject, since it is usually introduced in quantum-mechanics courses and textbooks in a way that differs substantially from the presentation employed in classical mechanics. In view of this disconnection, the understanding of this physical process by the student is particularly difficult.

In classical mechanics,[1,2] the description of the process of elastic scattering is performed through the study of the time evolution of a particle in the presence of a potential $V(r)$. In this presentation, the attention is focused on the orbit of each individual projectile and the evaluation of the *differential cross section* is achieved by geometrical considerations on each trajectory. This treatment can be adapted to quantum mechanics by means of a time-dependent wave packet description of the scattering process.[3] However, this kind of treatment, even though being a somewhat direct translation to the quantum-mechanical language of the standard classical presentation, is considered to be too arduous for a standard course on quantum mechanics. Therefore, in courses and textbooks[4–7] the scattering process is usually described within a time-independent formalism, where the stationary Schrödinger equation

$$H\Psi_{\mathbf{p}}(\mathbf{r}) = \frac{p^2}{2m}\,\Psi_{\mathbf{p}}(\mathbf{r})$$

is solved with the asymptotic condition

$$\Psi_{\mathbf{p}}(\mathbf{r}) \underset{r\to\infty}{\approx} \sqrt{n_0}\left(e^{i\mathbf{p}\cdot\mathbf{r}/\hbar} + f(p,\hat{\mathbf{p}}\cdot\hat{\mathbf{r}})\,\frac{e^{ipr/\hbar}}{r}\right). \tag{1}$$

Here, n_0 is an arbitrary normalization constant. The first term of this limit is usually interpreted as the incoming beam of particles with momentum $\mathbf{p}$, while the second one is said to represent a steady outgoing flux of scattered particles.[5] The squared modulus of the amplitude $f(p,\cos\theta)$ defines the differential cross section

$$\frac{d\sigma}{d\Omega} = |f(p,\cos\theta)|^2.$$

Historically, scattering theory developed around this line of reasoning, in analogy with the theory of bound states. Only later was this traditional approach properly justified within a time-dependent formalism. Nowadays, this alternative presentation is standard in courses on scattering theory[3,8]

and in many of the more advanced textbooks on quantum mechanics. There, the familiar treatment in terms of the stationary states $\Psi_{\mathbf{p}}(\mathbf{r})$ is presented not on its own but more as a computational technique within the broader framework of a time-dependent theory. The only drawback of this otherwise legitimate procedure is that its presentation might consume a large portion of time devoted to a first course in the general principles of quantum mechanics. In fact, while this presentation takes at least a full chapter of the textbooks on scattering theory,[8] a fistful of sentences (only five in Landau's text[4]) suffices to achieve the same goal within the traditional time-independent approach. Generally, the price to pay for this economy of time and effort is the student's suspicion than the basis for the time-independent formalism might be less than satisfactory.

The proverbial sagacious students immediately realize that a number of short cuts have been taken, leaving aside many unanswered questions. To begin with, they mistrust the entity granted to the scattering state $\Psi_{\mathbf{p}}(\mathbf{r})$, since, during the course, they have been told that a continuum wave function is not normalizable and thus cannot represent a real state. Furthermore, as it depends on only one space variable $\mathbf{r}$, they do not find it is easy to understand how it will describe a beam of particles, as claimed in a number of textbooks. Finally, the scattering state $\Psi_{\mathbf{p}}(\mathbf{r})$ represents a stationary situation, while the actual scattering process and its traditional classical description are clearly time dependent.[3]

A number of texts on quantum mechanics provide halfway descriptions of the time-dependent formalism to help the student in removing these objections.[6] These explanations are commonly intended to assimilate the quantum-mechanical formalism to the more familiar classical description of the scattering process. In relation with this line of reasoning, many authors[9–13] have presented the numerical simulation of the dispersion of a wave packet by a central force as an illustrative tool for the teaching of quantum scattering theory. However, it does not help in narrowing the gap with the traditional classical approach.

In this article, we discuss an equally valid alternative which consists, not in adapting the time-dependent classical presentation to the quantum-mechanical formalism, but in taking the reverse route. We present a classical description of the scattering process that is analogous to the time-independent formalism in quantum mechanics. In this way, the standard time-independent approach can be seen by the students as the natural translation into quantum-mechanical terms of the more familiar classical description. We consider that this presentation, not conveying large and tedious calculations, is particularly adequate as a first step towards the teaching of the quantum-mechanical scattering theory.

II. DEFINITION OF THE DIFFERENTIAL CROSS SECTION

Let us review the standard *idealization* of a typical collision experiment. A uniform flux $\mathbf{J}$ of identical structureless particles with momentum $\mathbf{p}$ impinges from infinity upon a target consisting of N fixed force centers. The number of particles $dI/d\Omega$ that are scattered per time unit and solid angle in a given direction is not an adequate quantity for describing the collision process. It is obviously dependent on the number N of particles in the target and the incident flux J. However, under suitable experimental conditions, we can assume that $dI/d\Omega$ is proportional to N and J. Therefore, the *differential cross section*, defined as the coefficient of proportionality between these extensive quantities

$$\frac{d\sigma}{d\Omega}=\frac{dI/d\Omega}{JN}$$

would be independent of any particular experimental set-up. In principle, this reduced quantity is completely determined by the potential energy $V(r)$ of the projectiles in the field of each force center. To satisfy the proportionality of $dI/d\Omega$ with J and N, which is an essential condition for the previous operational definition of the cross section, it is necessary that each projectile scatters separately off one force center at most. For this to be so, a number of requirements have to be met by the actual scattering experiment. First, the flux has to be small enough so that the incident particles do not interact among themselves. Furthermore, the target must be sufficiently thin and the scatterers so sparsely distributed as to avoid multiple or coherent scattering off more than one center of force. These basic conditions are almost invariably fulfilled in the great majority of the scattering experiments by gaseous atomic targets. Under these conditions, each collision process involves only one projectile and one force center. Sending a beam of projectiles against a target assembly consisting of a great number of scattering centers only amounts to repeating this elementary process many times. Under the additional assumption that both the scattering centers in the target and the projectiles in the incoming beam are uniformly distributed, this individual scattering process is always repeated with similar initial conditions. The initial momentum $\mathbf{p}$ for each of them is known, but the perpendicular vector distance $\boldsymbol{\rho}$, which separates the actual trajectory from that of frontal collision, is not fixed. Due to this lack of information on the microscopic details of the collision, we must analyze the problem in terms of an *ensemble*, where the individual scattering events are uniformly distributed over the impact parameter $\boldsymbol{\rho}$. Many textbooks fail to stress this obvious but important point: In a typical collision experiment, what is actually measured by repeating an elementary process many times, is the probability of the scattering in a given direction of just one *single* particle.

III. CLASSICAL TIME-INDEPENDENT DESCRIPTION OF THE SCATTERING PROCESS

The scattering of one particle by a force center can be described from two different points of view. First, we can study the classical orbit $\mathbf{r}=\mathbf{r}(\boldsymbol{\rho},t)$ of the particle in the potential $V(r)$ [or equivalently the time evolution of a wave packet $\phi=\phi(\boldsymbol{\rho},t)$ representing the state of the particle in a quantum-mechanical description]. On the other hand, we can base our reasoning on a stationary state describing the steady flow of the ensemble, as is standard in elementary courses on quantum mechanics.[4–7] We apply this latter approach to the **classical collision** of a particle by a force center. This analysis is very simple and helps to clarify the analogous development of the time-independent formalism in quantum mechanics, without burying it in long and tedious calculations.

We start with an ensemble of identical particles impinging upon a force center. Let us assume that, in the stationary situation, we already know the density of particles $n(\mathbf{r})$ in each point of space. In a quantum-mechanical description of the scattering process, this density is given by the squared modulus of the stationary state $\Psi_{\mathbf{p}}(\mathbf{r})$. On the other hand, in a classical description, this density can be easily evaluated

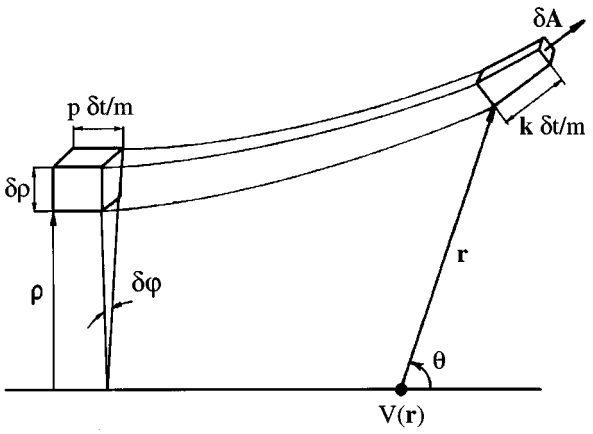

Fig. 1. Distortion of a control volume $\delta V(t)$ occupied by a fixed number of particles.

by studying the deformation suffered by a control volume—occupied by a fixed number of particles in the ensemble—due to the evolution of each particle in the presence of the potential $V(r)$. As is shown in the Appendix, the following simple expression is obtained:

$$n(\mathbf{r})=\frac{n_0}{\sin\,\theta}\sum_{\rho}\left|\left(\frac{\partial r}{\partial \rho}\right)_{\theta}\right|^{-1}, \tag{2}$$

where we have defined the initial density $n_0=J/(p/m)$. The coordinates r and θ are defined as shown in Fig. 1. The summation is carried out over all the impact parameters ρ for which there is a trajectory that goes through the point $\mathbf{r}$ (Fig. 2). In principle, in the asymptotic region, these contributions amount to a term n_+ representing the incident particles in the ensemble and a term n_- that accounts for the scattered particles moving outward from the force center

$$n(\mathbf{r}) \underset{r\to\infty}{\approx} n_+(\mathbf{r})+n_-(\mathbf{r}).$$

By definition, when $r\to\infty$ we have $n_+(\mathbf{r})\to n_0$. Moreover, the asymptotic limit of the outgoing term can be readily evaluated from Eq. (2) (see the Appendix)

$$n_-(\mathbf{r}) \underset{r\to\infty}{\approx} n_0\sum_{\rho_-}\frac{\rho}{r^2\,\sin\,\theta}\left|\frac{\partial\theta}{\partial\rho}\right|^{-1}, \tag{3}$$

where the impact parameters ρ_- characterize the outgoing trajectories through the point $\mathbf{r}=(r,\theta,\varphi)$. A similar asymptotic separation of the density in incoming and outgoing components also occurs in the quantum-mechanical description of the scattering process, but incorporating now an interference term between both contributions. From the asymptotic limit of the stationary scattering state, as given by Eq. (1), we obtain that, as in the classical case, $n_+(\mathbf{r})\to n_0$ when $r\to\infty$. The outgoing term is given by the squared modulus of the scattering amplitude

$$n_-(\mathbf{r}) \underset{r\to\infty}{\approx} \frac{n_0}{r^2}\,|f(p,\cos\,\theta)|^2. \tag{4}$$

Now we assume that only the outgoing contribution n_- can reach a detector located at a distance r from the force center. In actual experiments, any contribution from incoming particles (and interference terms) is avoided by an adequate collimation of the incident beam. Therefore, the number of particles detected per unit time and solid angle is given by $dI/d\Omega=r^2 n_-(\mathbf{r})k_r/m$, where the radial component k_r of the momentum equals p in the asymptotic limit. Finally, the following expression for the differential cross section is obtained:

$$\frac{d\sigma}{d\Omega}=\frac{dI/d\Omega}{J}=\lim_{r\to\infty} r^2\,\frac{n_-(\mathbf{r})}{n_0}.$$

By replacing $n_-(\mathbf{r})$ with Eqs. (3) and (4) the standard expressions for the scattering cross section are obtained in the classical

$$\frac{d\sigma}{d\Omega}=\sum_{\rho_-}\frac{\rho}{\sin\,\theta}\left|\frac{\partial\theta}{\partial\rho}\right|^{-1}$$

and quantum-mechanical descriptions

$$\frac{d\sigma}{d\Omega}=|f(p,\cos\,\theta)|^2.$$

IV. FINAL REMARKS

In many textbooks on quantum mechanics, the scattering process is generally presented by means of a time-independent approach, which enables a quicker and easier presentation than the standard time-dependent wave packet formalism. However, the student feels uneasy because of a number of loose ends. One of these drawbacks is that the necessary transition from the actual scattering experiment, with particles moving in the field of the target, to the idealization of a steady probability fluid is not always clearly explained. In this paper we have shown how a classical time-independent description of the scattering process can help in the understanding of these essential concepts. The calculations, as summarized in the Appendix, are so simple than they can even be performed as an exercise by the students themselves. In this way, the presentation of the standard time-independent formalism can be more easily achieved by basing it on an analogous classical description, rather than by relying upon a fast but untidy explanation of the time-dependent approach.

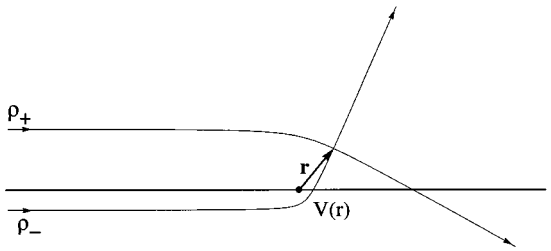

Fig. 2. Each point of space can be reached by more than one trajectory.

APPENDIX: CALCULATION OF THE DENSITY OF PARTICLES

Long before the collision, the number of particles in the control volume shown in Fig. 1 is $\delta N=n_0\cdot\delta V_0$, with $n_0=Jm/p$ and $\delta V_0=(p/m)\,\delta t\rho\,\delta\rho\,\delta\varphi$. As the particles approach the force center, this control volume changes to $\delta V(t)=\delta\mathbf{A}\cdot\mathbf{k}\delta t/m$, where $\mathbf{k}$ is the momentum of the particle at time t and $\delta\mathbf{A}=(r\,\sin\,\theta\delta\varphi\hat{l})\times(\partial\mathbf{r}/\partial\rho)_t\delta\rho$, with $\hat{l}=\hat{\rho}\times\hat{p}$, a

unit vector normal to the plane in Fig. 1. Therefore, the particle density inside this control volume is given by

$$n(\mathbf{r}) = \delta N/\delta V = \frac{n_0}{\sin\theta}\frac{\rho}{r}\left[\left(\hat{l}\times\left(\frac{\partial \mathbf{r}}{\partial \rho}\right)_t\right)\cdot\frac{\mathbf{k}}{p}\right]^{-1}$$

$$= \frac{n_0}{\sin\theta}\frac{\rho}{r}\left[\frac{\mathbf{k}}{p}\times\left(\frac{\partial \mathbf{r}}{\partial \rho}\right)_t\right]^{-1}.$$

Since we are under stationary flux conditions, we would like to eliminate all explicit dependence on the time variable t. By writing the cross product in polar coordinates it can be written in terms of the Jacobian of the transformation between the spatial coordinates r and θ and the parameters ρ and t of the orbit

$$n(\mathbf{r}) = \frac{n_0}{\sin\theta}\frac{\rho p}{m r^2}\left|\frac{\partial(r,\theta)}{\partial(\rho,t)}\right|^{-1}.$$

The implicit function theorem[14] lets us write this equation in terms of θ or $\dot{r}$. Then, after a little algebra and applying the conservation of angular momentum $l=\rho p = m r^2(\partial\theta/\partial t)_\rho$, the following alternative expressions can be obtained:

$$n(\mathbf{r}) = \frac{n_0}{\sin\theta}\frac{k}{k_r}\frac{\rho}{r^2}\left|\left(\frac{\partial\theta}{\partial\rho}\right)_r\right|^{-1},$$

$$n(\mathbf{r}) = \frac{n_0}{\sin\theta}\left|\left(\frac{\partial r}{\partial\rho}\right)_\theta\right|^{-1}.$$

For pedagogical reasons, we have been assuming that only one trajectory passes through any given point $\mathbf{r}$. Nevertheless, this is generally untrue and all possible contributions have to be added up as is done in Eqs. (2) and (3).

[a]Also member of the Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Argentina.

[b]Comisión Nacional de Energía Atómica y Universidad Nacional de Cuyo, Argentina.

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TAKING A NEW THEORY SERIOUSLY

It is not really so important to pinpoint the moment when physicists became 75% or 90% or 99% convinced of the correctness of general relativity. The important thing for the progress of physics is not the decision that a theory is true, but the decision that it is worth taking seriously—worth teaching to graduate students, worth writing textbooks about, above all, worth incorporating into one's own research.

Steven Weinberg, *Dreams of a Final Theory* (Pantheon Books, New York, 1992), p. 103.

THE FAON

My mood improved when Wally [Selove] and his team discovered a new particle. It was a heavy, neutral meson, and Wally was entitled to name it. He called it the f-zero, and it became known to a small circle of friends, and to all of my subsequent students, as the faon, after my name. Fortunately, the small circle of friends included Viki Weisskopf. When subsequent generations tried to bring some coherence into the naming of particles, and considered renaming the f-meson, Viki stuck up for the faon. It is a pleasure for me still to look up the *Particle Properties Data Booklet* used by physicists throughout the world, and find my particle there. It has now become the mother of a whole family of mesons.

Fay Ajzenberg-Selove, *A Matter of Choices—Memoirs of a Female Physicist* (Rutgers University Press, New Brunswick, New Jersey, 1994), p. 133.